Weighted Voronoi Diagrams in the L_∞-Norm

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Abstract

We study Voronoi diagrams of n weighted points in the plane in the maximum norm. We establish a tight $\Theta(n^2)$ worst-case combinatorial bound for such a Voronoi diagram and introduce an incremental construction algorithm that allows its computation in $\mathcal{O}(n^2 \log n)$ time.

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1 Introduction and Definition

In 1984 Aurenhammer and Edelsbrunner [?] introduced a worst-case optimal $\mathcal{O}(n^2)$ time algorithm to compute the Voronoi diagram of n multiplicatively weighted point sites in the L_2 metric. We investigate Voronoi diagrams of multiplicatively weighted point sites in the L_∞ metric. Contrary to the L_2 diagram, which consists of circular arcs, the L_∞ diagram is given by a PSGL. There is no obvious way to extend the linear-time half-space intersection of [?], which relies on a spherical inversion, to our setting, i.e., to scaled unit cubes.

Let S denote a finite set of n weighted points, *sites*, in $\mathbb{R}^2$ and consider a weight function $w\colon S \to \mathbb{R}^+$ assigning a weight $w(s)$ to every site. For the sake of descriptional simplicity we assume all weights of S to be unique. The weighted L_∞ distance $d_w(p,s)$ between an arbitrary point p in $\mathbb{R}^2$ and a site $s \in S$ is the standard L_∞ distance $d(p,s)$ between p and s divided by the weight of s. For s_i in S, the *(weighted) Voronoi region* $\mathcal{R}(s_i)$ is the set of all points of the plane that are closer to s_i than to any other site in S. The multiplicatively weighted Voronoi Diagram $\mathcal{V}^\infty(S)$ is a subdivision of the plane whose faces are given by (the connected components of) the Voronoi regions of all sites of S. The *bisector* of two distinct sites s_i, s_j of S models the set of points that are at the same weighted distance from s_i and s_j. Let $\Box_i(t)$ denote the boundary of an axis-aligned square centered at s_i with a side length of $2 \cdot t \cdot w(s_i)$. Let $\mathcal{U}(t)$ be the set of all such n unit squares scaled by t and corresponding weights. Let $\Box_i(t)$, $\Box_j(t)$ of $\mathcal{U}(t)$ and $w(s_i) < w(s_j)$. At time $t > 0$ these two squares intersect the first time and at time $t' > t$ $\Box_j(t)$ contains $\Box_i(t)$ for the first time. The bisector of s_i, s_j is traced out along $\Box_j(t) \cap \Box_j(t)$ between the times t and t'. A degree-two vertex, *joint*, in the bisector occurs whenever at least one vertex of one square crosses a side of another square. Since this can happen at most once for every vertex-side pair, the bisector of two sites forms a star-shaped polygon with a constant number of vertices.

Clearly $\mathcal{V}^\infty(S)$ is formed by portions of bisectors. Thus $\mathcal{V}^\infty(S)$ consists of straight-line segments and forms a PSLG. It contains Voronoi joints as vertices of degree two, and Voronoi nodes as vertices of degree higher than two. Note that our distinct-weight assumption prevents $\mathcal{V}^\infty(S)$ from containing unbounded edges: Let s_i be the site of S with maximum weight. Then there exists a time t_i such that $\Box_i(t)$ contains all other squares of $\mathcal{U}(t)$ for all $t > t_i$. Thus, the Voronoi region of s_i is the only unbounded region.

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2 Combinatorial Complexity of $\mathcal{V}^\infty(S)$ and Algorithm

Aurenhammer and Edelsbrunner [**?**] show that a multiplicatively weighted Voronoi diagram in the Euclidean metric has $\Theta(n^2)$ faces, edges, and nodes in the worst case. Their example that illustrates the quadratic worst-case lower bound can be adapted easily to our setting, hence establishing a quadratic lower bound for $\mathcal{V}^\infty(S)$ as well. Their proof of the quadratic upper bound proof is tightly connected to their setting and does not apply to $\mathcal{V}^\infty(S)$.

In the following we sketch how we establish a tight upper bound for $\mathcal{V}^\infty(S)$. The basic idea is that we raise $\mathcal{U}(t)$ to $\mathbb{R}^3$ by assigning a z-coordinate equal to t to every $\square_i(t)$. Then $\mathcal{U}(t)$, for $0 \le t \le \infty$, forms n upside-down pyramids whose apices lie on the xy-plane and coincide with their respective site. The slope of such a pyramid depends on the weight: A larger weight corresponds to smaller slope. Let $\widehat{\mathcal{U}}$ denote this pyramid arrangement. We can show that $\mathcal{V}^\infty(S)$ is the minimization diagram of $\widehat{\mathcal{U}}$.

Now let the sites of S be (re-)numbered such that $w(s_i) > w(s_j)$ for $1 \le i < j \le n$, and let $S_i := \{s_1, \ldots, s_i\}$. Hence, S_i contains all i sites of S with largest weights. We now focus on the combinatorial complexity of $\mathcal{V}^\infty(S)$. Suppose that one constructs the Voronoi region $\mathcal{R}(s_i)$ and merges it with $\mathcal{V}^\infty(S_{i-1})$ to obtain $\mathcal{V}^\infty(S_i)$. Similarly, in $\widehat{\mathcal{U}}$ we can add the respective pyramids incrementally such that $\widehat{\mathcal{U}}_i$ is the arrangement of all pyramids for S_i. We can show that the newly added pyramid P_i for s_i intersects at most a linear number of edges of the lower envelope of $\widehat{\mathcal{U}}_{i-1}$: Since the weight of s_i is smaller than the weights of all sites of S_i, all pyramids of $\widehat{\mathcal{U}}_{i-1}$ have sides with slopes that are smaller than the slope of the four sides of P_i. Now consider the supporting planes of the four sides of P_i. We look at the intersection of $\widehat{\mathcal{U}}_{i-1}$ and one such plane Π. We show that every pyramid of $\widehat{\mathcal{U}}_{i-1}$ forms a totally defined continuous function in this intersection and that any pair of these functions has the same value at most twice. This property helps to establish a linear upper bound on the combinatorial complexity of the lower envelope of $\Pi \cap \widehat{\mathcal{U}}_{i-1}$. Since all four such envelopes imply an overall linear bound we can conclude that inserting the pyramid P_i into $\widehat{\mathcal{U}}_{i-1}$ results in a linear number of edges in $\mathcal{R}(s_i)$, thus establishing the quadratic upper bound for $\mathcal{V}^\infty(S)$.

Next we sketch our incremental construction algorithm. The first site inserted is s_1 and initially $\mathcal{R}(s_1)$ is the xy-plane. In general, $\mathcal{R}(s_i)$ relative to S_i forms a star-shaped polygon with s_i in its kernel: As stated above, the bisector of two sites s_i,s_j, where $w(s_i) < w(s_j)$, forms a star-shaped polygon of constant combinatorial complexity around s_i. Hence, the intersection of these $i-1$ polygons that model the bisectors between s_i and all sites of S_{i-1} is again a star-shaped polygon with s_i in its kernel: It is $\mathcal{R}(s_i)$ relative to S_i. We can compute such a star-shaped polygon in $\mathcal{O}(n \log n)$ time using a simple divide&conquer approach. As established above, each such polygon is of at most linear size. Merging $\mathcal{V}^\infty(S_{i-1})$ with $\mathcal{R}(s_i)$ takes $\mathcal{O}(n \log n)$ time when utilizing a search structure that is at most quadratic in size; it holds the order of segments that lie on a common line. Finally we delete the edges of $\mathcal{V}^\infty(S_{i-1})$ that lie strictly in the interior of $\mathcal{R}(s_i)$. Let k_i be the number of edges of $\mathcal{V}^\infty(S_{i-1})$ strictly inside of $\mathcal{R}(s_i)$. Then $K := \sum_{0<i\le n} k_i \subseteq \mathcal{O}(n^2)$. This claim holds as K can be bounded by the number of edges created during the incremental construction, which in turn is bounded by the combinatorial complexity of $\mathcal{V}^\infty(S_i)$ which is in $\Theta(i^2)$.

▶ **Theorem 2.1.** *An incremental construction allows to compute $\mathcal{V}^\infty(S)$ of a set S of n weighted sites in $\mathcal{O}(n^2 \log n)$ time and $\mathcal{O}(n^2)$ space.*

References

1 F. Aurenhammer and H. Edelsbrunner. An Optimal Algorithm for Constructing the Weighted Voronoi Diagram in the Plane. *Pattern Recogn.*, 17(2):251 – 257, 1984.